Introduction to geodynamic modelling

Solving equations

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Goals of this lecture

- Give a quick overview of what we need to know to **solve equations**

- Provide a bit of background for why we need to use **numerical solutions** to equations

Finding a solution

- We have now seen the general form of several different equations we will be using to model geodynamic processes, such as the heat transfer equation below

$$\rho c_P \left(\frac{\partial T}{\partial t} + \boldsymbol{V} \cdot \nabla T \right) = \nabla \cdot k \nabla T + A$$

- But how do we go from this form of the equation to a solution in the form

$T(x, y, z, t) = \ldots?$

A simpler example

- Let's start simpler
  - Steady-state heat conduction in 1D

$$\rho c_P \left(\underbrace{\frac{\partial T}{\partial t}}_{\text{Time dependence}} + \underbrace{v_z \frac{\partial T}{\partial z}}_{\text{Advection}} \right) = \underbrace{\frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)}_{\text{Conduction}} + \underbrace{A}_{\text{Production}}$$

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Time dependence Advection Conduction Production

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Time dependence Advection Conduction Production

  - In this case we can ignore most terms of this equation as we're left with

$$\frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) = 0$$

A simpler example

- In fact, we can even factor out k and simplify our equation further

$$\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = 0$$
$$\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) = 0$$
$$\frac{\partial^2 T}{\partial z^2} = 0$$

- Here, we are left with simply the second derivative of temperature T with respect to depth z

- We can solve this, we just need to integrate twice with respect to z

A simpler example...?

- We can integrate once to find

$$\int \frac{d^2T}{dz^2} dz = \frac{dT}{dz} + c_1$$

$$\frac{dT}{dz} + c_1 = 0$$

- And a second time to get

$$\int \left(\frac{dT}{dz} + c_1 \right) dz = \int \frac{dT}{dz} dz + \int c_1 dz$$
$$= T(z) + c_1 z + c_2$$

$$\boxed{\begin{aligned} T(z) + c_1 z + c_2 &= 0 \\ T(z) &= -c_1 z - c_2 \end{aligned}}$$

OK, now what?

$$T(z) = -c_1 z - c_2$$

- Great, we have our solution, but we can't yet calculate temperatures because we don't know the values of c_1 and c_2, the constants of integration
  - What do we need to find c_1 and c_2?

Boundary conditions

- With two integrations we end up with two unknown constants
- In order to find the values of the constants, we need to use **boundary conditions**, known (or assumed) values for certain variables in our equation
- There are two natural choices for our heat transfer problem:
  - Known temperatures and certain depths
  - Known temperature gradients at certain depths

Boundary conditions

$$T(z) = -c_1 z - c_2$$

- Let's make some assumptions
  - We know the surface temperature at z = 0 is T = 0
  - We know the temperature at some depth z = L is T = 1000°C
- If we substitute in assumption 1, we find $c_2 = 0$
- If we then substitute in assumption 2, we find $c_1 = -1000 / L$

Boundary conditions

- Now, we can finally see out exciting result with c_1 = -1000 / L and c_2 = 0…

$$T(z) = -c_1 z - c_2$$

$$T(z) = \frac{1000}{L} z$$

a straight line.

Other considerations

- So that was an almost embarrassingly simple example
  - Heat conducted between two known temperatures at steady state will simply produce a linear temperature increase with depth
- What happens, though, if we consider a non-steady state example?
  - What other information might we need to know for that case?

Initial conditions

Time-dependent advection and conduction

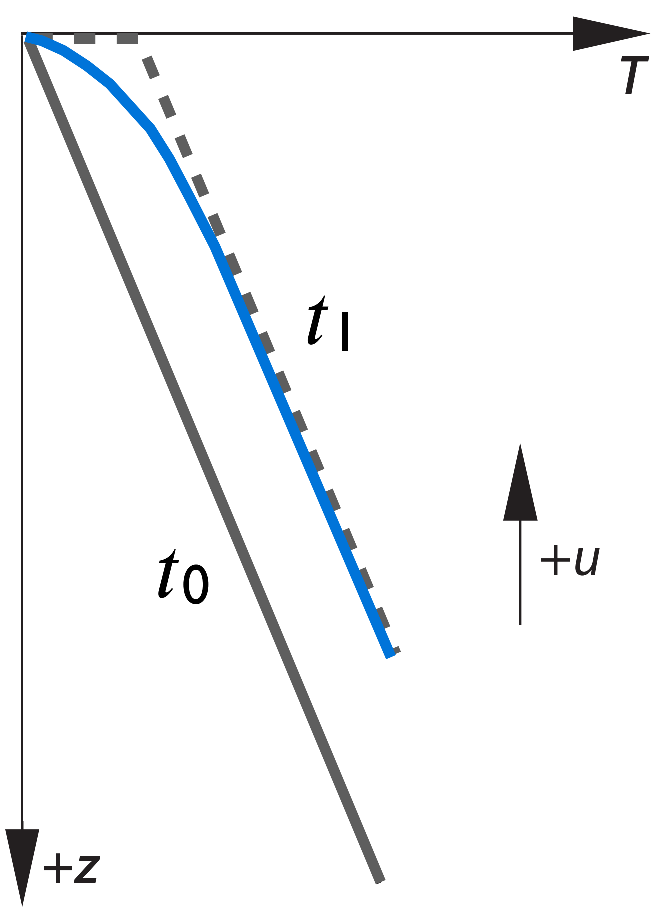

Fig. 3.13, Stüwe, 2007

- In order to consider the time evolution of the temperatures in the Earth, we would also need to know the starting distribution of temperature
  - This is known as an **initial condition**
- Clearly, the temperatures we expect after some time can strongly depend on the initial conditions we assume
  - Why don't we need initial conditions for the steady-state case?

What about other scenarios

- What if we want to solve slightly more complicated problems?
  - 2D heat transfer with surface topography
  - Spatially variable material properties
  - Temperature-dependent material properties

What about other scenarios

- What if we want to solve slightly more complicated problems?
  - 2D heat transfer with surface topography
  - Spatially variable material properties
  - Temperature-dependent material properties
- In most of these cases we cannot simply directly integrate our heat transfer equation because we cannot find the constants of integration
  - Consider the example above with topography
  - We likely do not have a function that can give us the required known values of temperature and elevation at the surface

The need for numerical integration

- In cases where the geometry of the problem or material property distributions/behaviors are more complex, we need to use numerical methods to integrate our equations of interest

- Much days 2 and 3 in this course will focus on the use of the **finite difference method** of solving equations, how it can be used, and its limitations

- The finite difference method is one of two popular approaches used for solving equations in geodynamic models
    - We'll start learning the details of the finite difference method tomorrow morning

Summary

- To solve equations used in geodynamic modelling (or anything really) we need to know
  - **Boundary conditions**
  - **Initial conditions** (in some cases)

- We can solve these equations directly in some cases, but most of the time we will need to use numerical methods such as the **finite difference method** to find solutions

References

Stüwe, K. (2007). *Geodynamics of the lithosphere: an introduction*. Springer Science & Business Media.